

# A New Method for Numerical Abel-Inversion

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A new numerical solution of Abel-type integral equations is presented which is based on the principle of Fourier-analysis. The unknown radial distribution is expanded in a series of cosine-functions, the amplitudes of which are calculated by least-squares-fitting of the Abel-transformed series to the measured data. Advantages and applications of the new method are discussed, followed by a short comparison with methods, commonly used for Abel-inversion.

## Problem

In the study of radially symmetrical objects (see Fig. 1, e.g. a plasma column) the radial distribution  $f(r)$  of some physical quantity (e.g. particle density) very often cannot be measured directly, but only through a function  $h(y)$  which is connected with  $f(r)$  by the integral equation

$$h(y) = 2 \int_y^R f(r) \frac{r}{\sqrt{r^2 - y^2}} dr, \quad (1)$$

which can be inverted analytically [1]:

$$f(r) = -\frac{1}{\pi} \int_r^R \frac{dh(y)}{dy} \frac{1}{\sqrt{y^2 - r^2}} dy. \quad (2)$$

In practice,  $f(r)$  is calculated either by discretizing (1) or (2) and using pre-calculated matrix-elements [2, 3], or by using (2) and representing the measured curve  $h(y)$  by analytical functions, e.g. [4]. Frequently problems are caused by measurement-errors and, especially, by the differentiation of the noisy data  $h(y)$  in (2).

## New Method

In rough analogy with a method developed for the case of asymmetry [5], the unknown distribution  $f(r)$  is expanded in a series similar to a Fourier-series:

$$f(r) = \sum_{n=N_l}^{N_u} A_n f_n(r) \quad (3)$$

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with unknown amplitudes  $A_n$ , where  $f_n(r)$  is a set of cosine-functions, e.g.

$$f_0(r) = 1, \quad f_n(r) = 1 - (-1)^n \cos\left(n\pi \frac{r}{R}\right). \quad (4)$$

Following (1), the Abel-transform  $H(y)$  of (3) has the form

$$H(y) = 2 \sum_{n=N_l}^{N_u} A_n \int_y^R f_n(r) \frac{r}{\sqrt{r^2 - y^2}} dr, \quad (5)$$

each integral

$$h_n(y) = \int_y^R f_n(r) \frac{r}{\sqrt{r^2 - y^2}} dr \quad (6)$$

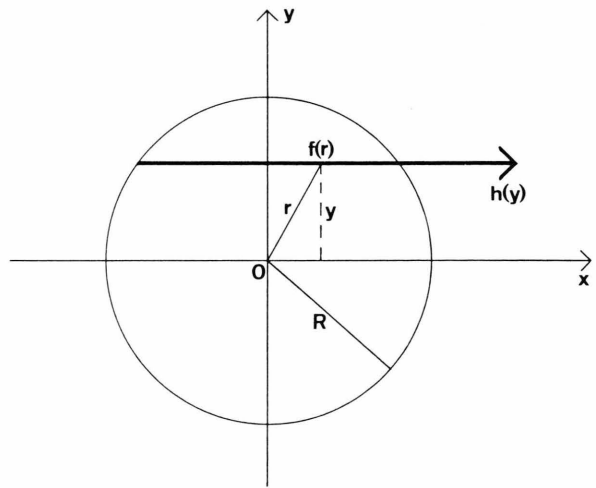


Fig. 1. The radial distribution  $f(r)$  cannot be measured directly, but only through the integral  $h(y)$  in  $x$ -direction.

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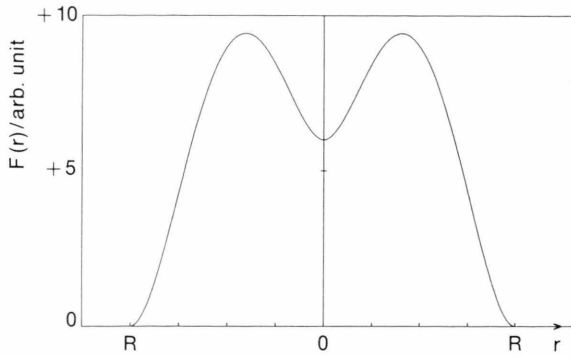


Fig. 2. Example for an analytical function  $F(r)$  used for computer simulated comparison.

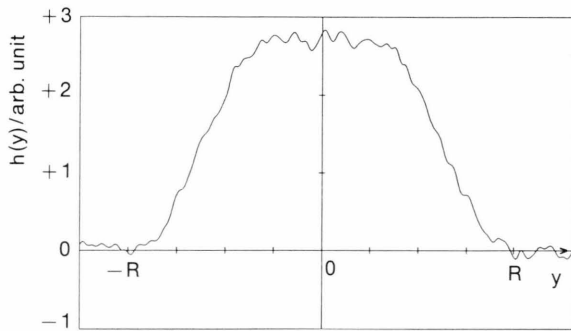


Fig. 3. Abel-transform of  $F(r)$  superimposed with statistical noise.

having been integrated numerically in advance and stored on a special file. Equation (5) is least squares fitted to the real data  $h(y)$  at each of the  $N$  measured points  $y_k$ :

$$\sum_{k=1}^N (H(y_k) - h(y_k))^2 \rightarrow \min, \quad (7)$$

leading to

$$\sum_{n=N_l}^{N_u} A_n \sum_{k=1}^N (h_n(y_k) h_m(y_k)) = \sum_{k=1}^N (h(y_k) h_m(y_k)), \quad (8)$$

$$\forall m: N_l \leq m < N_u.$$

The equation system (8) yields the amplitudes  $A_n$  which are inserted into (3), thus producing the resulting distribution  $f(r)$ .

### Advantages of the Proposed Method

- The method is derivative-free.
- Neither smoothing nor any other pre-treating of the measured data  $h(y)$  is necessary.

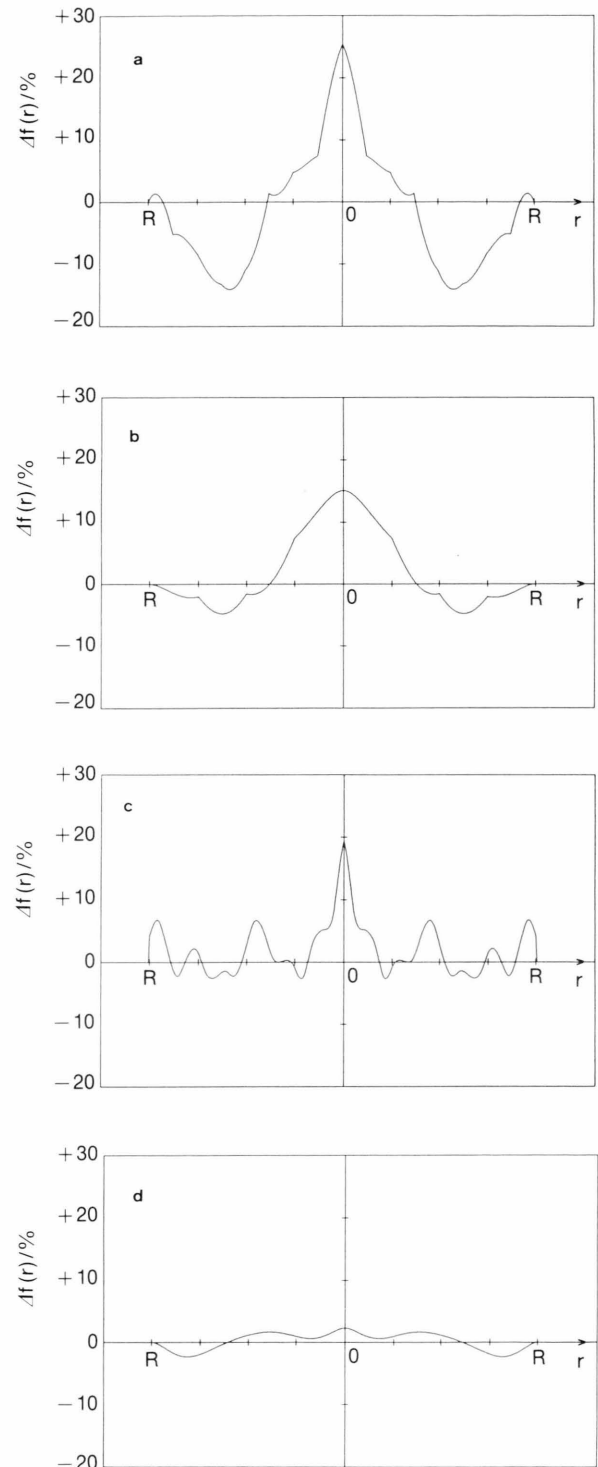


Fig. 4. Difference of function  $F(r)$  (Fig. 2) from distribution  $f(r)$ , calculated using (a) matrix method [3]; (b) spline-interpolation for  $h(y)$  [4]; (c) one-dimensional convolution method [6]; (d) proposed new method.

- The exact form of the functions  $f_n(r)$  can be adapted to the given physical conditions.
- The method can be used as a noise-filter by choosing the lower and upper frequency-limits  $N_l$  and  $N_u$ .

### Comparison by Computer Simulation

An analytical function  $F(r)$  (e.g. Fig. 2) is Abel-transformed (1) and superimposed with statistical noise (Figure 3). The result, simulating the measured data  $h(y)$ , is the base for calculating  $f(r)$  using different methods. Figure 4 shows the deviation of the calculated distributions  $f(r)$  from the starting function  $F(r)$ .

### Conclusion

The new method is non-iterative, derivative-free, and adaptable to any special problem of this kind, hence its accuracy is superior to the compared methods for almost all tested profiles (example see Fig. 4), specially with high noise-level. So this Fourier-method can be recommended for all practical applications of Abel-inversion.

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